ON THE APPLICATION OF THE BUBNOV-GALERKIN METHOD IN THE THEORY OF HYDRODYNAMIC STABILITY

(O PRIMENENII METODA BUBNOVA-GALERKINA V TEORII GIDRODINAMICHESKOI USTOICHIVOSTI)

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As is well known [1], the problem of the stability of plane-parallel flows (or those which are nearly so) of a viscous, incompressible fluid reduces to the problem of the eigenvalues of the Orr-Sommerfeld equation,

$$L\varphi = \varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi - i\alpha R[(u-c)(\varphi''-\alpha^2\varphi) - u''\varphi] = 0 \tag{1}$$

The boundary conditions for flow with two solid boundaries are

$$\varphi(0) = \varphi'(0) = \varphi(1) = \varphi'(1) = 0 \quad (0 \leqslant x \leqslant 1) \tag{2}$$

for flow with one solid boundary (boundary layer) are

$$\varphi(0) = \varphi'(0) = 0, \ |\varphi(x)| < M = \text{const} \quad (0 \leqslant x < +\infty) \tag{3}$$

and for flow without solid boundaries (free shear layer) are

$$|\varphi(x)| < M = \text{const} \quad (-\infty < x < +\infty) \tag{4}$$

Here α and R are positive numbers, u is a function of x (the velocity profile) which, for boundary conditions (3) and (4) tends to a constant value for $x \to +\infty$ (and also for $x \to -\infty$ in case (4)), c is a complex parameter with respect to which the eigenvalue problem is posed, for given α and R. An amplified disturbance corresponds to $c_i > 0$, where c_i is the imaginary part of the eigenvalue c, and a damped disturbance corresponds to $c_i < 0$; for neutral oscillations, $c_i = 0$. The value of R for the onset of instability is not large for certain flows, for example, if the velocity profile has a point of inflection [1]. In those cases, it may be expected that the method of Bubnov-Gelerkin will give good results for approximations that are acceptable in practice. However, the question of the convergence of this method as applied to Equation (1) has, up to now, been resolved only for boundary conditions (2.). The first proof of the convergence of the method was given by Petrov in 1940 [2]. In the present paper, the convergence of the Bubnov-Galerkin method is proved for Equation (1) with boundary conditions (3) and (4).

We shall investigate values of c in only that region D of the complex plane defined by the inequality $c_i > -\alpha/R$. The reasons for this restriction will be evident from the proof. In solving Equation (1), one is usually interested in the eigenvalues c, which correspond to amplified and neutral oscillations. All such c are found in region D. We shall replace the requirement for boundedness of the function $\varphi(x)$ in cases (3) and (4) by the equivalent condition

$$\varphi^{(k)}(x) \in L_{2(a,\,b)} \qquad (k = 0, 1, 2, 3, 4) \tag{5}$$

Here, $a = 0$ and $b = +\infty$ in case (3), and $a = -\infty$ and $b = +\infty$ in case (4). (The equivalence can be demonstrated by studying the behavior of the solution of (1) for large values of the argument). In what follows, we shall take the region of definition of L to be those functions $\varphi(x)$ which satisfy condition (5) and, in addition, the condition $\varphi(0) = \varphi'(0) = 0$ for $a = 0$.

Let

$$A\varphi = \varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi$$

For any two functions φ and ψ in the region of definition of the operator L, we introduce a scalar multiplication according to Formula

$$[\varphi, \psi] = (A\varphi, \psi)$$

where the parantheses denote a scalar product in $L_{2(a,b)}$. The operator A is symmetric positive definite, and thus the scalsr product which has been introduced satisfies all the requirements for a scalar product in Hilbert space [3]. The Hilbert space with the above scalar product, made complete in the usual way, will be denoted by H. It is easy to prove that, for arbitrary $\varphi \in H$ and $\psi \in H$, we may write

$$[\varphi, \psi] = (\varphi'' - \alpha^2\varphi, \psi'' - \alpha^2\psi) \tag{6}$$

Equation (1) is equivalent to Equation

$$L_1\varphi = A^{-1}L\varphi = 0 \tag{7}$$

For φ and ψ, we have, in the domain of definition of the operator L,

$$(L\varphi, \psi) = (AA^{-1}L\varphi, \psi) = [A^{-1}L\varphi, \psi] = [L_1\varphi, \psi] \tag{8}$$

To apply the Bubnov-Galerkin method, we take a system of linearly independent functions φ_k $(k = 1, 2, \ldots.)$ in the domain of definition of the operator L. We seek an approximate eigenfunction of Equation (1) in the form $\varphi = a_1\varphi_1 + \ldots + a_n\varphi_n$. The equations of the Bubnov-Galerkin method have the form

$$\left\{L \sum_{k=1}^{n} a_k\varphi_k, \varphi_i\right\} = 0 \qquad (i = 1, \ldots, n) \tag{9}$$

where the unknowns are the coefficients a_k $(k = 1, \ldots, n)$. The approximate eigenvalues are the roots of the determinant of the system of equations (9).

Making use of the equality (8), we rewrite the system (9) in the form

$$\left[L_1 \sum_{k=1}^{n} a_k\varphi_k, \varphi_i\right] = 0 \qquad (i = 1, \ldots, n) \tag{10}$$

Equations (10) are the Bubnov-Galerkin equations for Equation (7) in space H. It is easy to show that the operator L_1, is bounded in H. Let us extend it into all of H continuously. The sufficient conditions for convergence of the Bubnov-Galerkin method for linear equations with bounded operator were developed in [4]. These may be summarized as follows (Theorems 1, 4 and 5): let φ_k $(k = 1, 2, \ldots)$ be a system which is complete in H and, for every value c of region D, let the inequality

$$\inf \lim |[L_1\psi_n, \psi_n]| > 0 \quad \text{for} \quad n \to \infty$$

be satisfied for $n \to \infty$ for any sequence, ψ_n $(n = 1, 2, \ldots)$, $\|\psi_n\|_H = 1$, in H which converges weakly to zero. Then, any bounded, closed set $D_0 \subset D$, not containing eigenvalues of Equation (7), starting with some n, will also not contain approximate eigenvalues, and, for any arbitrary eigenvalue $c \in D$ there exists a sequence of approximate eigenvalues converging to that c. The convergence of the method for computing the eigenfunctions is investigated in [5]. Let ψ_n $(n = 1, 2, \ldots)$ be any sequence of functions in H, $\|\psi_n\|_H = 1$, weakly convergent to zero. Let us investigate the real part,

$$\mathrm{Re}\,[L_1\psi_n, \psi_n] = (\psi_n'' - \alpha^2\psi_n, \psi_n'' - \alpha^2\psi_n) - \alpha Rc_i(\psi_n'' - \alpha^2\psi_n, \psi_n) +$$
$$+ \alpha R\,\mathrm{Im}\,(u\psi_n'', \psi_n) = 1 + \alpha Rc_i(\|\psi_n'\|^2 + \alpha^2\|\psi_n\|^2) - \alpha R\,\mathrm{Im}\,(\psi_n', u'\psi_n) \tag{11}$$

In deriving Equations (11), integration by parts and Equation (6) were used. $\|\psi\|$ denotes the norm of the function ψ in $L_{2(a,b)}$. Let

$$f_n = \psi_n'' - \alpha^2\psi_n$$

The weak convergence to zero in H of the sequence $\{\psi_n\}$ means that the sequence $\{f_n\}$ converges weakly to zero in $L_{2(a,b)}$, while the condition $\|\psi_n\|_H = 1$ may be written in the form $\|f_n\| = 1$. Integrating by parts, we obtain Equation $\|\psi_n'\|^2 + \alpha^2\|\psi_n\|^2 = -(f_n, \psi_n)$, from which, according to the Cauchy-Buniakowski inequality, $\|\psi_n'\|^2 + \alpha^2\|\psi_n\|^2 \leqslant \|f_n\|\|\psi_n\|$, and, consequently,

$$\|\psi_n\| \leqslant \alpha^{-2}\|f_n\| = \alpha^{-2}, \qquad \|\psi_n'\|^2 + \alpha^2\|\psi_n\|^2 \leqslant \alpha^{-2}\|f_n\|^2 = \alpha^{-2} \tag{12}$$

Let us express ψ_n in terms of f_n. We have

$$\psi_n = \int_a^b K(x, y) f_n(y)\, dy \tag{13}$$

where

$$K(x, y) = \begin{cases} -{}^1/_2\alpha^{-1}e^{-\alpha(x-y)} & \text{for } x \geqslant y \\ -{}^1/_2\alpha^{-1}e^{\alpha(x-y)} & \text{for } x \leqslant y \end{cases}$$

in the case where $a = -\infty$, $b = +\infty$, and

$$K(x, y) = \begin{cases} -{}^1/_2\alpha^{-1}e^{-\alpha(x-y)} + {}^1/_2\alpha^{-1}e^{-\alpha(x+y)} & \text{for } x \geqslant y \\ -{}^1/_2\alpha^{-1}e^{\alpha(x-y)} + {}^1/_2\alpha^{-1}e^{-\alpha(x+y)} & \text{for } x \leqslant y \end{cases}$$

in the case where $a = 0$, $b = +\infty$. Formula (13) can be verified directly. Furthermore,

$$u'\psi_n = \int_a^b K_1(x, y) f_n(y)\, dy \quad (K_1(x, y) = u'(x) K(x, y)) \tag{14}$$

The function u' usually tends to zero sufficiently fast with $|x| \to \infty$ so that

$$\int_a^b (u')^2\, dx$$

exists. Making use of this, it is easy to show that

$$\int_a^b\int_a^b |K_1(x, y)|^2\, dxdy < +\infty$$

Therefore, the integral operator given by the kernel $K_1(x,y)$ in Equation (14) is completely continuous operator in $L_{2(a,b)}$. In view of the well known properties of completely continuous operators, it follows, from the weak convergence of the sequence $\{f_n\}$ to zero in $L_{2(a,b)}$, that the sequence $\{u'\psi_n\}$ converges strongly to zero in $L_{2(a,b)}$. Since the sequence $\{\psi_n'\}$ is bounded in $L_{2(a,b)}$, because of (12), it follows that $\lim(\psi_n', u'\psi_n) = 0$ for $n \to \infty$. Using this equality, we obtain from Equation (11)

$$\inf \lim_{n\to\infty} \operatorname{Re}[L_1\psi_n, \psi_n] = \inf \lim_{n\to\infty} \{1 + \alpha Rc_i (\|\psi_n'\|^2 + \alpha^2\|\psi_n\|^2)$$

From this and inequality (12),

$$\inf \lim_{n\to\infty} \operatorname{Re}[L_1\psi_n, \psi_n] \geqslant 1 \qquad \text{for } c_i \geqslant 0$$

$$\inf \lim_{n\to\infty} \operatorname{Re}[L_1\psi_n, \psi_n] \geqslant 1 + \alpha^{-1}Rc_i \qquad \text{for } c_i \leqslant 0$$

Consequently,

$$\inf \lim_{n\to\infty} |[L_1\psi_n, \psi_n]| > 0, \qquad c \in D$$

To ensure convergence of the method, it is necessary to select a system of approximating functions $\{\varphi_k\}$, from the region of definition of operator L, which is complete in H. This can be done, for example, as follows.

Take a system of functions $\{g_k\}$ which is complete in $L_{2(a, b)}$. Then it is easy to show that the system $\{\varphi_k = A^{-1}g_k\}$ is complete in H. However, such a choice for the system may make the calculation somewhat cumbersome. The systems $x^n e^{-1/2 x^2}$ and $x^{n+2} e^{-1/2 x}$ $(n = 0, 1, 2, \ldots)$ have sufficiently simple form for calculating the integrals for cases (4) and (3), respectively. If they are orthogonalized, then we obtain

$$e^{1/2 x^2} \frac{d^n e^{-x^2}}{dx^n}, \quad e^{1/2 x} x^{-1/2 \beta} \frac{d^n}{dx^n} (x^{n+\beta} e^{-x}) \quad (\beta = 4, n = 0, 1, 2, \ldots)$$

i.e. the well-known Hermite functions and Laguerre functions.

The completeness in H of these systems follows from theorems given in [6].

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